Exercise 1. Consider the system

$$X' = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} X$$

where $\lambda_1 < \lambda_2 < \lambda_3 < 0$ Describe how the solution through an arbitrary initial value tends to the origin.

Solution. The system is a sink. All solutions tends exponentially to the origin. □

Exercise 2. Find the general solution of

$$X' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & -3 & -1 \\ 0 & 3 & -2 & -2 \\ 1 & 1 & -2 & 0 \end{pmatrix} X$$

Describe the stable and unstable space. Draw a phase diagram for the 2 spaces.

Solution. The characteristic polynomial is given by

$$(1-\lambda)(-\lambda^3 + \lambda^2 - 2) = 0$$

hence the roots are $1, -1, 1+i$ and $1-i$. Thus the stable space has dimension 1 and the unstable space has dimension 3. The phase portrait is pictured in figure 1 looks as follows with all lines flowing outwards. The phase portrait for the 1-dimensional stable space is trivial. □

Exercise 4. Consider the system

$$X' = \begin{pmatrix} a & 0 & b \\ 0 & b & 0 \\ -b & 0 & a \end{pmatrix} X$$

which depends on two real parameters a and b. Find the general solution of this system. Sketch the regions in the ab-plane where the system has different types of phase planes.

Solution. The characteristic polynomial is given by

$$(a-\lambda)(\lambda^2 - \lambda ab + b^2 + a^2) = 0$$

Hence the eigenvalues are $\lambda_1 = b$ and $\lambda_\pm = a \pm ib$. If b is not equal to zero, then all eigenvalues are distinct and hence the system is diagonalizeable. If b is equal to zero then system is diagonalizeable by inspection. Thus the general solution is given by

$$X(t) = c_1 v_1 e^{bt} + e^{at} \cdot \left(c_+ v_+ e^{ibt} + c_- v_- e^{-ibt}\right)$$

□

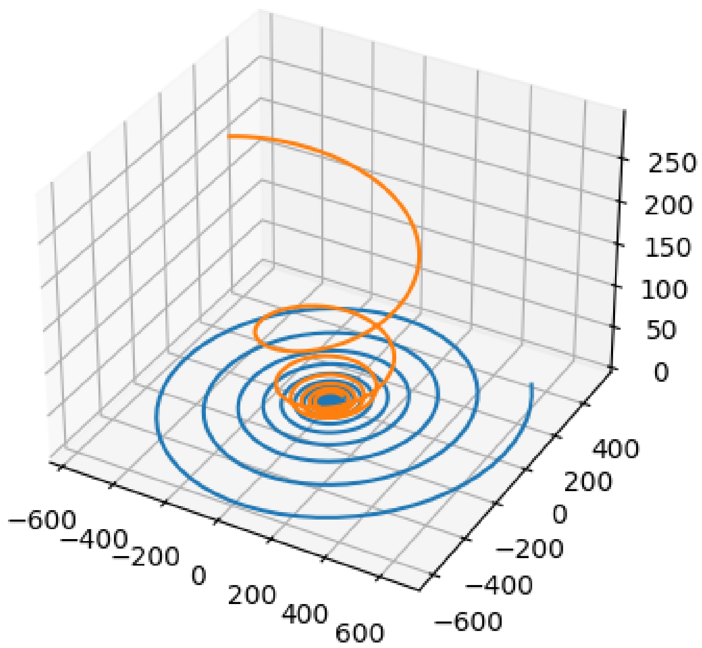

Figure 1: Phase portrait for exercise 3.

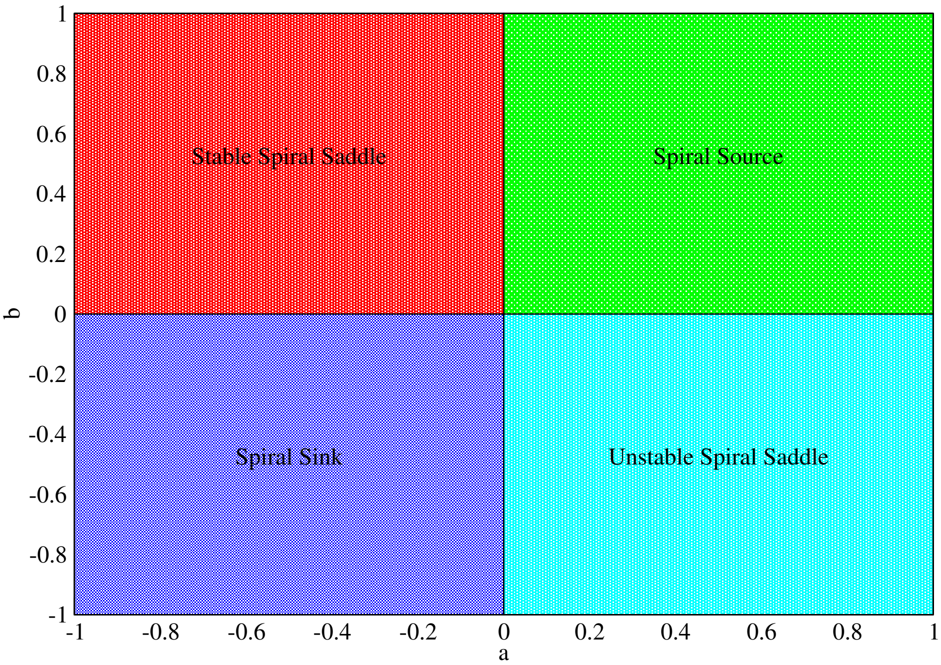

Figure 2: ab-plane for exercise 3.