Exercise 11.5:2. Use integration by parts to verify that

$$\mathcal{L}[t](s) = \int_0^\infty te^{-st}dt = \frac{1}{s^2} \qquad \text{for } s > 0$$

Solution.

$$\int_0^\infty te^{-st}dt \lim_{a\to\infty}\int_0^a te^{-st}dt = \lim_{a\to\infty}\left[t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_{t=0}^{t=a} = \lim_{a\to\infty}\frac{1}{s^2} - \frac{a}{se^{sa}} - \frac{1}{s^2e^{sa}} =$$

By L'hopitals rule,

$$\lim_{a\to\infty}\frac{a}{se^{sa}} = \lim_{a\to\infty}\frac{1}{s^2e^{sa}} = 0$$

thus

$$\mathcal{L}[t](s) = \frac{1}{s^2}$$

□

Exercise 11.5:8. Compute $\mathcal{L}[\cos(t+a)](s)$.

Solution. Applying integration by parts twice we see that

$$\int \cos(t+a)e^{-st}dt = -\cos(t+a)\frac{1}{s}e^{-st} + \sin(t+a)\frac{1}{s^2}e^{-st} - \int \cos(t+a)\frac{1}{s^2}e^{-st}dt$$

hence

$$\int \cos(t+a)e^{-st}dt = \frac{-\cos(t+a)\frac{1}{s}e^{-st} + \sin(t+a)\frac{1}{s^2}e^{-st}}{\left(1+\frac{1}{s^2}\right)} = e^{-st}\frac{-\cos(t+a)+\sin(t+a)\frac{1}{s}}{s+\frac{1}{s}}$$

therefore

$$\mathcal{L}[\cos(t+a)](s) = \int_0^\infty \cos(t+a)e^{-st}dt = \frac{\cos(a)-\sin(a)\frac{1}{s}}{s+\frac{1}{s}} + \lim_{t\to\infty} e^{-st}\frac{-\cos(t+a)+\sin(t+a)\frac{1}{s}}{s+\frac{1}{s}}$$

By a straight forward application of the squeeze theorem, the limit converges to zero, hence

$$\mathcal{L}[\cos(t+a)](s) = \frac{\cos(a)-\sin(a)\frac{1}{s}}{s+\frac{1}{s}} = \frac{s\cos(a)-\sin(a)}{s^2+1}$$

□

Exercise 11.5:12. Find the inverse Laplace transform of

$$\frac{2}{s^2+4}$$

.

Solution. I claim that

$$\mathcal{L}(\sin(bt)) = \frac{b}{s^2+b^2}$$

Let us verify; Applying integration by parts twice we see that

$$\int \sin(bt)e^{-st}dt = -\sin(bt)\frac{1}{s}e^{-st} - b\cos(bt)\frac{1}{s^2}e^{-st} - \int b^2\sin(bt)\frac{1}{s^2}e^{-st}dt$$

hence

$$\int \sin(bt)e^{-st}dt = -\frac{\sin(bt)\frac{1}{s}e^{-st} + b\cos(bt)\frac{1}{s^2}e^{-st}}{1+\frac{b^2}{s^2}} = -e^{-st}\frac{\sin(bt) + b\cos(bt)\frac{1}{s}}{s+\frac{b^2}{s}}$$

so

$$\mathcal{L}(\sin(bt)) = \int_0^\infty \sin(bt)e^{-st}dt = \frac{b\frac{1}{s}}{s+\frac{b^2}{s}} + \lim_{t\to\infty} -e^{-st}\frac{\sin(bt) + b\cos(bt)\frac{1}{s}}{s+\frac{b^2}{s}}$$

By a straight forward application of the squeeze theorem, the limit converges to zero, hence

$$\mathcal{L}(\sin(bt)) = \frac{b\frac{1}{s}}{s+\frac{b^2}{s}} = \frac{b}{s^2+b^2}$$

□

Exercise 11.5:24. Let $P(D)$ be an nth-order linear constant coefficient, differential operator. Show that

$$\mathcal{L}[P(D)y](s) = P(s)\mathcal{L}[y](s) + Q(s)$$

for some polynomial Q of degree $n-1$. Use induction.

Solutions. The proof is by induction. For the basecase let $n=1$ then $P(D)y = ay' + by$ for some constants a, b. First remark that by integration by parts we have

$$\int ye^{-st}dt = -\frac{y}{s}e^{-st} + \frac{1}{s}\int y'e^{-st}dt$$

therefore

$$s\mathcal{L}[y] - \mathcal{L}[y'] = \lim_{x\to\infty}\left[-y(t)e^{-st}\right]_{t=0}^{t=x} = y(0) - \lim_{x\to\infty} y(x)e^{-sx}$$

Assuming that the Laplace transforms on the left hand side exists, then the limit on the right hand side converges for all $s > 0$. It follows that $\lim_{x\to\infty} y(x)e^{-sx} = 0$ thus

$$\mathcal{L}[Dy] = \mathcal{L}[y'] = s\mathcal{L}[y] - y(0) \qquad (\star)$$

So therefore

$$\mathcal{L}[P(D)y](s) = a\mathcal{L}[y'](s) + b\mathcal{L}[y](s) = a\left(s\mathcal{L}[y] - y(0)\right) + b\mathcal{L}[y](s) = P(s)\mathcal{L}[y] - ay(0)$$

Since $s \mapsto ay(0)$ is polynomial of degree zero, then this completes the basecase.

For the induction step, let $P(D)$ be a polynomial of degree $n+1$, then by the fundamental theorem of algebra $P(D) = (aD+b)\circ(\tilde{P}(D))$ for some polynomial $\tilde{P}$ of degree atmost n, so

$$\mathcal{L}[P(D)y](s) = \mathcal{L}[aD\circ\tilde{P}(D)y + b\tilde{P}(D)y](s) = a\mathcal{L}[D\circ\tilde{P}(D)y] + b\mathcal{L}[\tilde{P}(D)y](s)$$

and by $(\star)$ we have

$$\mathcal{L}[D\circ\tilde{P}(D)y] = s\mathcal{L}[\tilde{P}(D)y] - (\tilde{P}(D)y)(0)$$

so

$$\begin{aligned}\mathcal{L}[P(D)y](s) &= a\left(s\mathcal{L}[\tilde{P}(D)y](s) - (\tilde{P}(D)y)(0)\right) + b\mathcal{L}[\tilde{P}(D)y](s)\\ &= (as+b)\,\mathcal{L}[\tilde{P}(D)y](s) - a\cdot(\tilde{P}(D)y)(0)\end{aligned}$$

By the induction hypothesis

$$\mathcal{L}[\tilde{P}(D)y](s) = \tilde{P}(s)\mathcal{L}[y](s) + Q(s)$$

for some polynomial Q of degree $n-1$. Therefore

$$\begin{aligned}\mathcal{L}[P(D)y](s) &= (as+b)\cdot\Big(\tilde{P}(s)\mathcal{L}[y](s) + Q(s)\Big) - a\cdot(\tilde{P}(D)y)(0)\\ &= P(s)\mathcal{L}[y](s) + (as+b)\cdot Q(s) - a\cdot(\tilde{P}(D)y)(0)\end{aligned}$$

Since Q is polynomial of degree $n-1$ then $s \mapsto (as+b)\cdot Q(s) - a\cdot(\tilde{P}(D)y)(0)$ is a polynomial of degree n. This completes the induction step. □

Exercise 11.6:2. Compute the convolution

$$t^2 * (t^2+1)$$

Solution.

$$\begin{aligned}t^2 * (t^2+1) &= \int_0^t u^2\cdot((t-u)^2+1)du = \int_0^t u^2\cdot(t^2+u^2-2ut+1)du\\ &= \int_0^t u^2t^2 + u^4 - 2u^3t + u^2 du\\ &= \left[\frac{u^3}{3}t^2 + \frac{u^5}{5} - \frac{2u^4}{4}t + \frac{u^3}{3}\right]_{u=0}^{u=t}\\ &= \frac{t^3}{3}t^2 + \frac{t^5}{5} - \frac{t^4}{2}t + \frac{t^3}{3}\\ &= t^5\left(\frac{10}{30} + \frac{6}{30} - \frac{15}{30}\right) + \frac{t^3}{3}\\ &= \frac{t^5}{30} + \frac{t^3}{3}\end{aligned}$$

□

Exercise 11.6:8. Using convolution compute the inverse Laplace transform of

$$\frac{1}{(s^2+1)(s-1)}$$

Solutions.

$$\frac{1}{(s^2+1)(s-1)} = \frac{1}{(s^2+1)}\frac{1}{(s-1)} = \mathcal{L}[\sin(t)](s)\mathcal{L}[e^t](s) = \mathcal{L}[\sin(t) * e^t]$$

so it suffices to compute the convolution

$$\sin(t) * e^t = \int_0^t \sin(u)\cdot e^{t-u}du = e^t\int_0^t \sin(u)\cdot e^{-u}du$$

Applying integration by parts twice, we see that

$$\int \sin(u)e^{-u}du = -\sin(u)e^{-u} - \cos(u)e^{-u} - \int \sin(u)e^{-u}du$$

so

$$\int_0^t \sin(u) \cdot e^{-u} du = \left[\frac{-1}{2} \left(\sin(u) e^{-u} + \cos(u) e^{-u} \right) \right]_{u=0}^{u=t} = \frac{-1}{2} \left(\left(\sin(t) + \cos(t) \right) e^{-t} - 1 \right)$$

therefore

$$\sin(t) * e^t = \frac{-1}{2} \left(\sin(t) + \cos(t) - e^t \right) = \frac{1}{2} \left(e^t - \sin(t) - \cos(t) \right)$$

□

Exercise 11.6:10. Compute the inverse Laplace transform of

$$\frac{e^{-2s}}{s(s^2+4)}$$

Solution. First we notice that

$$\begin{aligned} \frac{e^{-2s}}{s(s^2+4)} &= e^{-2s} \frac{1}{s} \frac{1}{s^2+4} \\ &= e^{-2s} \mathcal{L}\left[\frac{1}{2}\right](s) \mathcal{L}[\sin(2t)](s) \\ &= e^{-2s} \mathcal{L}\left[\frac{1}{2} * \sin(2t)\right](s) \end{aligned}$$

Since

$$\frac{1}{2} \int \sin(2t - 2u) du = \frac{1}{4} \cos(2t - 2u)$$

then

$$\frac{1}{2} * \sin(2t) = \frac{1}{2} \int_0^t \sin(2t - 2u) du = \frac{1}{4} \left(\cos(0) - \cos(2t) \right) = \frac{1}{4} \left(1 - \cos(2t) \right)$$

therefore

$$\frac{e^{-2s}}{s(s^2+4)} = e^{-2s} \mathcal{L}\left[\frac{1}{4}\left(1 - \cos(2t)\right)\right](s) = \mathcal{L}\left[\frac{1}{4}\left(1 - \cos(2(t-2))\right)\right](s)$$

so the inverse Lapalce transform is

$$\frac{1}{4} \left(1 - \cos(2t - 4) \right)$$

□

Exercise 11.6:22. Recall that the gamma function can be defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \qquad z > 0$$

(a) Use integration by parts to show that

$$\Gamma(z+1) = z\Gamma(z)$$

(b) deduce from part a that $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \ldots$.
(c) Show that if $a > -1$ then

$$\mathcal{L}[t^a](s) = \frac{\Gamma(a+1)}{s^{a+1}}$$

Solution. By integration by parts we see that

$$\int t^z e^{-t} dt = -t^z e^{-t} + z \int t^{z-1} e^{-t} dt$$

therefore

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt = \left[-t^z e^{-t}\right]_{t=0}^{t=\infty} + z\Gamma(z)$$

hence the result follows once we show that $[-t^z e^{-t}]_{t=0}^{t=\infty} = 0$. To this end

$$\left[-t^z e^{-t}\right]_{t=0}^{t=\infty} = \left(\lim_{t\to\infty} -t^z e^{-t}\right) + 0^z \cdot e^0 = 0$$

where the limit can be evaluated (for fixed z) using L'hopitals rule a finite number of times.

(b) We prove it by induction; First the basecase $n = 0$;

$$\Gamma(0+1) = \int_0^\infty t^{1-1} e^{-t} dt = \int_0^\infty e^{-t} dt = 1$$

Now suppose that $\Gamma(n+1) = n!$ then by part (a)

$$\Gamma((n+1)+1) = (n+1)\Gamma(n+1) = (n+1) \cdot n! = (n+1)!$$

(c) Let $u = st$ then $du = sdt$ so

$$\mathcal{L}[t^a](s) = \int_0^\infty e^{-st} t^a dt = \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^a du = \frac{1}{s^{a+1}} \Gamma(a+1)$$

□