Exercise 11.1:12. Determine the general solution to the following differential equation. Then determine the specific solution

$$y'' - 3y - y = 0 \qquad \text{with} \qquad y(0) = 0 \text{ and } y(1) = 0$$

Solution. Let $x(t) = y'(t)$ then

$$\begin{aligned} x'(t) &= 3x(t) + y(t) \\ y'(t) &= x(t) \end{aligned}$$

Concisely this can be written as

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

We compute the eigenvalues

$$\det(A - \lambda_\pm I) = \det \begin{pmatrix} 3-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = -\lambda(3-\lambda) - 1 = \lambda^2 - 3\lambda - 1 = 0$$

so

$$\lambda_\pm = \frac{3}{2} \pm \frac{\sqrt{13}}{2}$$

Since the eigenvalues are distinct, then A can be diagonalized. By definition an eigenvector, is any vector satisfying $Av_\pm = \lambda_\pm v_\pm$ so $v_\pm \in \operatorname{Ker}(A - \lambda_\pm I)$ hence

$$v_\pm = \begin{pmatrix} \lambda_\pm - 3 \\ 1 \end{pmatrix}$$

are eigenvectors. Now by diagonalization the general solution is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_+ v_+ e^{\lambda_+ t} + c_- v_- e^{\lambda_- t} = e^{\frac{3}{2}t} \cdot \begin{pmatrix} c_+(-\frac{3}{2} + \frac{\sqrt{13}}{2})e^{\frac{\sqrt{13}}{2}t} + c_-(-\frac{3}{2} - \frac{\sqrt{13}}{2})e^{-\frac{\sqrt{13}}{2}t} \\ c_+ e^{\frac{\sqrt{13}}{2}t} + c_- e^{-\frac{\sqrt{13}}{2}t} \end{pmatrix}$$

We now determine a specific solution. Since $y(0) = 0$ then $c_+ + c_- = 0$ so $c_- = -c_+$. Since $y(1) = 0$ then

$$c_+ \left(e^{\frac{\sqrt{13}}{2}} - e^{-\frac{\sqrt{13}}{2}} \right) = 0$$

so $c_+ = 0$. □

Exercise 11.2A:18. Determine the general solution to the following differential equation. Then determine the specific solution

$$y'' - y' + y = 0 \qquad \text{with} \qquad y(0) = 2 \text{ and } y'(0) = -1$$

Solution. Let $x(t) = y'(t)$ then

$$\begin{aligned} x'(t) &= x(t) - y(t) \\ y'(t) &= x(t) \end{aligned}$$

Concisely this can be written as

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

We compute the eigenvalues

$$\det(A-\lambda_\pm I)=\det\begin{pmatrix}1-\lambda & -1\\ 1 & -\lambda\end{pmatrix}=-\lambda(1-\lambda)+1=\lambda^2-\lambda+1=0$$

so

$$\lambda_\pm=\frac{1}{2}\pm\frac{\sqrt{-3}}{2}=\frac{1}{2}\pm\frac{\sqrt{3}i}{2}$$

Since the eigenvalues are distinct, then A can be diagonalized over $\mathbb{C}$. By definition an eigenvector, is any vector satisfying $Av_\pm=\lambda_\pm v_\pm$ so $v_\pm\in\operatorname{Ker}(A-\lambda_\pm I)$ hence

$$v_\pm=\begin{pmatrix}\lambda_\pm\\ 1\end{pmatrix}$$

are eigenvectors. Now by diagonalization the general solution is given by

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=c_+v_+e^{\lambda_+t}+c_-v_-e^{\lambda_-t}$$

Using standard trigonmetric identities we see that

$$y(t)=e^{\frac{1}{2}t}\left((c_++c_-)\cos\left(\frac{\sqrt{3}}{2}t\right)+(c_+-c_-)i\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

Since $y(0)=2$ then $c_++c_-=2$. Now $x(0)=-1$ so

$$-1=(c_+)\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)+c_-\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)=(c_++c_-)\frac{1}{2}+(c_+-c_-)\frac{\sqrt{3}i}{2}$$

hence

$$(c_+-c_-)=\frac{4i}{\sqrt{3}}$$

thus

$$y(t)=e^{\frac{1}{2}t}\left(2\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{4}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

□

Exercise 11.2A:46. Let $y(x)$ be a solution of a second-order equation $y''+ay'+by=0$. Can constants a and b be chosen so that $y(x)=x\cos(x)$? What about the function $y(x)=\sin(x)+\cos(2x)$? Justify your answers.

Solution. Suppose $y(x)=x\cos(x)$ then

$$\begin{aligned}y(x)&=x\cos(x)\\ y'(x)&=\cos(x)-x\sin(x)\\ y''(x)&=-2\sin(x)-x\cos(x)\end{aligned}$$

so

$$0=y''(x)+ay'(x)+by=-2\sin(x)-x\cos(x)+a\left(\cos(x)-x\sin(x)\right)+bx\cos(x)$$

for all x. If $x = 0$ then $a = 0$. On the otherhand if $x = \frac{\pi}{2}$ then

$$-2 - a\left(\frac{\pi}{2}\right) = 0$$

so $a = -\frac{4}{\pi} \neq 0$. Therefore y cannot be solution to the second-order equation $y'' + ay' + by = 0$.

Suppose $y(x) = \sin(x) + \cos(2x)$ then

$$\begin{aligned} y(x) &= \sin(x) + \cos(2x) \\ y'(x) &= \cos(x) - 2\sin(2x) \\ y''(x) &= -\sin(x) - 4\cos(2x) \end{aligned}$$

so

$$0 = y''(x) + ay'(x) + by(x) = -\sin(x) - 4\cos(2x) + a\left(\cos(x) - 2\sin(2x)\right) + b\left(\sin(x) + \cos(2x)\right)$$

for all x. Consider $x = \frac{\pi}{2}$ then

$$y''(x) + ay'(x) + by(x) = -1 + 4 = 3 \neq 0$$

Therefore y cannot be solution to the second-order equation $y'' + ay' + by = 0$. □

Exercise 11.2B:20. Find real constant coefficient linear differential equation of the smallest possible order that has the function $y(x)$ as solution.

$$y(x) = x\cos(4x)$$

Proof. The smallest possible order is 4. To see this rewrite the equation in to a system of linear differential equations. First we observe that the system must have atleast 1 purely imaginary eigenvalue and that this eigenvalue must be repeated. But complex eigenvalues comes in conjugated pairs, and as such the degree must be atleast 4.

$$\begin{aligned} y(x) &= x\cos(4x) \\ y'(x) &= \cos(4x) - 4x\sin(4x) \\ y''(x) &= -8\sin(4x) - 16x\cos(4x) \\ y'''(x) &= -48\cos(4x) + 64x\sin(4x) \\ y''''(x) &= -256\sin(4x) + 256x\cos(4x) \end{aligned}$$

From which it is easy to see that

$$y''''(x) + 32 \cdot y''(x) + 256y(x) = 0$$

□