Exercise 3.4:21. Define a function G from $C(\mathbb{R})$ to $C(\mathbb{R})$ by

$$G(u)(x) = \int_0^x tu(t)dt$$

Show that G is linear. Show that G is one-to-one. Describe the image under G of the subspace of $\mathcal{P}_n$ consisting of all polynomials of degree atmost n. Describe the image under G of all of $\mathcal{P}$. Describe the inverse of G. Find an element of $\mathcal{P}$ that is not in the image of G.

Solution. Let $\alpha, \beta \in \mathbb{R}$ and let $u, v \in C(\mathbb{R})$ then

$$G(\alpha u + \beta v) = \int_0^x t(\alpha u(t) + \beta v(t))dt = \alpha \int_0^x tu(t)dt + \beta \int_0^x v(t)dt = \alpha G(u) + \beta G(v)$$

so G is linear. Suppose $G(u) = 0$ then

$$\int_0^x tu(t)dt = 0$$

for every $x \in \mathbb{R}$ hence $u(t) \equiv 0$. So $\text{Ker}(G) = 0$ so G is injective. Let $u = \sum_{k=0}^n a_k x^k$ then

$$G(u)(x) = \int_0^x \sum_{k=0}^n a_k t^{k+1} dt = \sum_{k=0}^n a_k \frac{x^{k+2}}{k+2} = \sum_{k=0}^n \frac{a_k}{k+2} x^{k+2}$$

so the image consists of polynomials of the form $v = \sum_{k=2}^{n+2} a_k x^k$. For all polynomials similar reasons gives that the image consists of all polynomials $\sum_{k=0}^N a_k x^k$ with $a_0 = a_1 = 0$.

The problem is stupid because there is no such thing as "the" inverse since G is not surjective. But as it is injective there is a left sided inverse. Let f be in the image of G, then f is C^1, $f(0) = 0$ and by the fundamental theorem of calculus $f'(x) = 0$ so let $u_f(x) = \frac{f'(x)}{x}$ for $x \neq 0$ and $u_f(0) = 0$ then

$$G(u) = \int_0^x tu_f(t)dt = \int_0^x f'(x)dt = f(x) - f(0) = f(x)$$

So define a left sided inverse $F : C(\mathbb{R}) \to C(\mathbb{R})$ by $F(f) = u_f$ for f in the image of G and $F(f) = 0$ for f not in the image of G.

Finally notice that $f(x) = 1$ is certainly in $\mathcal{P}$, but

$$G(u)(0) = \int_0^0 tu(t)dt = 0$$

so f is not in the image. □

Exercise 3.5B:4. Show that the given set of vectors forms a basis for $\mathbb{R}^3$ of dimension 3 by showing (a) spanning, and (b) independence.

$$\{(1, 2, 3), (0, 0, 1), (2, 2, 4)\}$$

Solution. It suffices to show that the 3 standard basis vectors of $\mathbb{R}^3$ are in the span of $\{(1, 2, 3), (0, 0, 1), (2, 2, 4)\}$. Clearly e_3 is in the span, thus

$$(1, 2, 0) \text{ and } (2, 2, 0)$$

are also in the span so

$$(1,0,0) = (2,2,0) - (1,2,0)$$

is in the span, but then

$$(0,1,0) = \frac{1}{2}\left((1,2,0) - (1,0,0)\right)$$

which completes the proof. Independence is obvious by dimension counting. □

Exercise 3.5B:15. Find the dimension of the subspaces of $\mathbb{R}^2$ spanned by the given vectors.

$$\{(-1,1),(1,-1)\}$$

Solution. Since $-1(-1,1) = (1,-1)$ then the two vectors are linear dependent and since neither is zero, then the dimension of the subspace spanned is 1. □

Exercise 3.5B:41. Let S be the shift operator defined on the vector space $\mathcal{P}_2$ of polynomials of degree at most 2 by $S(p)(x) = p(x+1)$. (a) Show that S is a linear operator. Show that $S = I + D + \frac{1}{2}D^2$, where $D = \frac{d}{dx}$, and I stands for the identity operator. (c) Show that on $\mathcal{P}_n$ the space of polynomials of degree at most n, the shift operator is related to D by $S = 1 + D + \frac{1}{2!}D^2 + \cdots + \frac{1}{n!}D^n$.

Solution. (a) Let $\alpha, \beta \in \mathbb{R}$ and $p, q \in \mathcal{P}_2$ then

$$S(\alpha p + \beta q) = \alpha p(x+1) + \beta q(x+1) = \alpha S(p) + \beta S(q)$$

so p, q are linear. (b) it suffices to check the idenity on a basis so

$$S(1) = 1 = I(1) + D(1) + \frac{1}{2}D^2(1)$$
$$S(x) = x + 1 = I(x) + D(x+1) + \frac{1}{2}D^2(x+1)$$
$$S(x^2) = (x+1)^2 = x^2 + 2x + 1 = I(x^2) + D(x^2) + \frac{1}{2}D^2(x+1)$$

More generally by the binomial theorem we have

$$S(x^k) = (x+1)^k = x^k + kx^{k-1} + \frac{k\cdot(k-1)}{2}x^{k-1} + \cdots + \frac{k!}{k!}x^{k-k}$$

□

Exercise 3.5C:3. Show that if f is a one-to-one linear function, then the set $\{f(x_1), ..., f(x_k)\}$ is linearly independent if and only if $\{x_1, ..., x_k\}$ is linearly independent. What does this imply about the dimensions of the image and domain of f?

Solution. Suppose that

$$\alpha_1 f(x_1) + \cdots + \alpha_k f(x_k) = 0$$

then by linearity

$$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) = 0$$

since f is injective then $\ker(f) = \{0\}$ so

$$\alpha_1 x_1 + \cdots + \alpha_k x_k = 0$$

hence by linearly independence of $\{x_1, ..., x_k\}$ we have $\alpha_1 = \cdots = \alpha_k = 0$.

Conversely suppose that

$$\alpha_1 x_1 + \cdots + \alpha_k x_k = 0$$

then

$$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) = f(0) = 0$$

then by linearity

$$\alpha_1 f(x_1) + \cdots + \alpha_k f(x_k) = 0$$

Since $\{f(x_1), ..., f(x_k)\}$ is linearly independent then $\alpha_1 = \cdots = \alpha_k = 0$.

This implies that if the vector space is finite dimensional and f is injective, then the dimension of of the image of f equals the dimension of the domain. □

Exercise 3.5C:10. Assume V and W are finite dimensional and that $f : V \to W$ is linear. Prove that if N is the kernel of f, then there is a subspace S of V such that $S \pitchfork N$, $S \cap N = \{0\}$, and f restricted to S is one-to-one.

Solution. Let n be the dimension of V and let $\{v_1, \ldots, v_k\}$ be a basis for N and extend it with $\{v_{k+1}, \ldots, v_n\}$ to a basis for V. Then

$$S = \text{span}(\{v_{k+1}, \ldots, v_n\})$$

Suppose $w \in S \cap N$, then there exists $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_{n-k}$ such that

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = w = \beta_1 v_{k+1} + \cdots + \beta_{n-k} v_n$$

hence

$$\alpha_1 v_1 + \cdots + \alpha_k v_k - \beta_1 v_{k+1} - \cdots - \beta_{n-k} v_n = 0$$

Since $v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$ is a basis for V, then they are linearly independent so

$$\alpha_1 = \cdots = \alpha_k = \beta_1 = \cdots = \beta_{n-k} = 0$$

hence $w = \vec{0}$ so $S \cap N = \{0\}$. Now $\text{Ker}(f|_S) = S \cap \text{Ker}(f) = S \cap N = \{0\}$ so $f|_S$ is injective. □

Exercise 3.6A:4. Find all the eigenvalues of each of the linear operators defined by the following matrices, and for each eigenvalue find an associated eigenvector.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$$

Solution. The eigenvalues are given by

$$\det \begin{pmatrix} -\lambda & 1 \\ 0 & (2-\lambda) \end{pmatrix} = -\lambda(2-\lambda)$$

so $\lambda_1 = 0$ and $\lambda_2 = 2$. Since

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so $v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Likewise

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

so $v_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ □

Exercise 3.6A:13. Find all the eigenvalues of

$$G = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

show that the associated eigenvectors span $\mathbb{R}^2$, and describe the action of G, as in example 2.

Solution. The eigenvalues are given by

$$\det \begin{pmatrix} 1-\lambda & 2 \\ 1 & (1-\lambda) \end{pmatrix} = (1-\lambda)^2 - 2$$

so $\lambda_+ = 1+\sqrt{2}$ and $\lambda_- = 1-\sqrt{2}$. Since the eigenvalues are real and distinct, then the associated eigenvectors span $\mathbb{R}^2$. The action of G is simply that it expands vectors along v_+ by a factor of $1+\sqrt{2}$ and contracts vectors along v_- by $1-\sqrt{2}$. □

Exercise 3.6A:14. Solve the system of differential equations using eigenvalues and eigenvectors as in Example 5 in the text.

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \vec{x}$$

Solution. The eigenvalues are given by

$$\det \begin{pmatrix} 1-\lambda & 4 \\ 1 & (1-\lambda) \end{pmatrix} = (1-\lambda)^2 - 4$$

so $\lambda_+ = 1+2 = 3$ and $\lambda_- = 1-2 = -1$. Since

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

then $v_+ = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and since

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

then $v_- = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ so the general solution is given by

$$\vec{x}(t) = c_+ v_+ e^{3t} + c_- v_- e^{-t} = \begin{pmatrix} 2c_+ e^{3t} - 2c_- e^{-t} \\ c_+ e^{3t} + c_- e^{-t} \end{pmatrix}$$

for some constants c_+ and c_-. □

Exercise 3.6B:4. Find all the eigenvalues of the linear operators defined by the following matrix and state whether or not Theorem 6.7 guarantees a basis of eigenvectors. If not determines if a basis of eigenvectors exists

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Solution.

$$\det(A - \lambda I) = 1 - \lambda^3$$

so the eigenvalues are the 3 cubic roots of 1. Since only 1 of these roots are real then A is not diagonalizeable over $\mathbb{R}$. Treating A as a complex matrix, then all the eigenvalues are distinct so A is diagonalizeable over $\mathbb{C}$. (Remark: Being diagonalizeable over $\mathbb{R}$ (respectively $\mathbb{C}$ is the same has having a basis for $\mathbb{R}^n$ (respectively $\mathbb{C}^n$) consisting of eigenvectors) □

Exercise 3.6B:8. Show that the matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

has complex associated independent complex eigenvaules. Also find associated independent complex eigenvectors.

Solution.

$$\det(R_\theta - \lambda I) = (\cos\theta - \lambda)^2 + (\sin\theta)^2 = 0$$

so $\lambda_\pm = \cos\theta \pm i\sin\theta$. Notice that

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \begin{pmatrix} \cos\theta \mp i\sin\theta \\ \sin\theta \pm i\cos\theta \end{pmatrix} = (\cos\theta \mp i\sin\theta) \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

so the eigenvectors are given by $v_\pm = \begin{pmatrix} 1 \\ \mp i \end{pmatrix}$. □

Exercise 3.7A:10. Let V be a 2-dimensional vector space with an inner product and a basis $\{u, v\}$, and let $(u, u) = a$, $(u, v) = b$, and $(v, v) = c$. (a) Let $x = pu + qv$ and $y = ru + sv$ be vectors in V. Use additivity and homogeneity of the inner product to show that

$$\langle x, y\rangle = \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix}$$

(b) Show that $a > 0$ and $c > 0$ and that the Cauchy-Schwarz inequality implies that $b^2 < ac$ (c) Show that if a, b and c satisfy the conditions of part (b) and (x, y) is defined by the formula in part (a) then (x, y) satisfies the conditions for being an inner product. [Hint: To show positivity, write out (x, x) in terms of a, b, c, p and q and use the technique of completing the square.]

Solution.

$$\begin{aligned} \langle x, y\rangle &= \langle pu + qv, ru + sv\rangle = pr\langle u, u\rangle + qs\langle v, v\rangle + (ps + rq)\langle u, v\rangle \\ &= pra + qsc + (ps + rq)b = \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \end{aligned}$$

Recall that the Cauchy-Schwarz inequality states

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle$$

with equality if and only if $x = \lambda y$ for some scalar λ. Using our computation form above we have

$$(pra + qsc + (ps + rq)b)^2 \leq (p^2a + q^2c + (2pq)b) \cdot (r^2a + s^2c + (2rs)b)$$

Taking $p = s = 1$ and $r = q = 0$ we see that

$$b^2 < ac$$

For (c) bilinearity and symmetry is clear from the definition of matrix multiplication. It remains to check positivity

$$\begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = p^2a + q^2c + (2pq)b$$

so positivity would follow if we show that whenever $(p, q) \neq (0, 0)$ then

$$2pq|b| < p^2a + q^2c$$

Recall that

$$0 \leq (p\sqrt{a} - q\sqrt{c})^2 = p^2a + q^2c - 2pq\sqrt{ac}$$

so

$$2pq\sqrt{ac} \leq p^2a + q^2c$$

since $b^2 < ac$ then $|b| \leq \sqrt{ac}$ so

$$2pq|b| < 2pq\sqrt{ac} \leq p^2a + q^2c$$

□

Exercise 3.7B:2. The vectors $(1, 1, 1)$ and $(1, 2, 1)$ span a plane $\mathcal{P}$ in $\mathbb{R}^3$. Use the Gram-Schmidt process to find an orthogonal basis for $\mathbb{R}^3$ in which the first two vectors form an orthogonal basis for $\mathcal{P}$.

Solution. First we compute a orthogonal basis. Let $u_1 = (1, 1, 1)/\sqrt{3}$ and

$$\begin{aligned} v_2 &= (1, 2, 1) - ((1, 2, 1) \bullet u_1) \cdot u_1 \\ &= (1, 2, 1) - \left((1, 2, 1) \bullet (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\right) \cdot (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \\ &= (1, 2, 1) - \frac{4}{\sqrt{3}}(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \\ &= (1, 2, 1) - (\frac{4}{3}, \frac{4}{3}, \frac{4}{3}) = (\frac{-1}{3}, \frac{2}{3}, \frac{-1}{3}) \end{aligned}$$

then we normalise

$$u_2 = \frac{v_2}{||v_2||} = \frac{3}{2}(\frac{-1}{3}, \frac{2}{3}, \frac{-1}{3}) = (\frac{-1}{2}, 1, \frac{-1}{2})$$

□