Exercise 3.2:12. Let $\vec{a}$ be a fixed nonzero vector in $\mathbb{R}^n$. (a) Show that the set S of all vectors x such that $a \bullet x = 0$ is a subspace of $\mathbb{R}^n$. (b) Show that if k is a nonzero real number, then the set A of all vectors x such that $a \bullet x = k$ is not a subspace.

Solution. (a) Let $\alpha, \beta \in \mathbb{R}$ and $x_1, x_2 \in S$. Then

$$a \bullet (\alpha x_1 + \beta x_2) = \alpha \cdot (a \bullet x_1) + \beta \cdot (a \bullet x_2) = \alpha \cdot 0 + \beta \cdot 0 = 0$$

so $(\alpha x_1 + \beta x_2) \in S$ hence S is a subspace.

(b) Since $a \bullet 0 = 0 \neq k$ then $0 \notin A$ so A is not a subspace. □

Exercise 3.2:16. In Exercises 16 to 18, determine whether the set of all polynomials p in $\mathcal{P}_3$ that satisfy the given conditions is a subspace of $\mathcal{P}_3$.

$$p(0) = 1$$

Solution. No. If $p_0 \equiv 0$ is the zero polynomial then $p(0) \neq 1$. □

Exercise 3.2:17.

$$p(1) = 0$$

Solution. Yes. Let $\alpha, \beta \in \mathbb{R}$ and $p, q \in \mathcal{P}_3$ such that $p(1) = q(1) = 0$. Then

$$\alpha \cdot p(1) + \beta q(1) = \alpha \cdot 0 + \beta \cdot 0 = 0$$

□

Exercise 3.2:18.

$$p(0) = p(1)$$

Solution. Yes. Let $\alpha, \beta \in \mathbb{R}$ and $p, q \in \mathcal{P}_3$ such that $p(0) = p(1)$ and $q(0) = q(1)$. Then

$$(\alpha \cdot p + \beta \cdot q)(1) = \alpha \cdot p(1) + \beta q(1) = \alpha \cdot p(0) + \beta q(0) = (\alpha \cdot p + \beta \cdot q)(0)$$

□

Exercise 3.2:20. In the space $\mathcal{P}$ of polynomials, let A be the set of all p such that $p(x) = -p(-x)$, and let B be the set of p such that $p(x) = p(-x)$. Show that A is the span of $\{x, x^3, x^5, ...\}$, and find a spanning set for B.

Recall that two polynomials are equal (have the same value at every $x \in \mathbb{R}$) if and only if they are identical (exact same coefficients in each and every power of x). This follows from the fact that a non-zero polynomial has at most finitely many roots, so if p, q are equal polynomials then $p - q$ has infinitely many roots and thus must be the zero polynomial.

Solution. Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial in A then $p(x) = -p(-x)$ so

$$\sum_{k=0}^{n} a_k x^k = -\sum_{k=0}^{n} a_k (-x)^k = \sum_{k=0}^{n} (-1)^{k+1} a_k x^k$$

Since the two polynomials are equal then $a_k = (-1)^{k+1} a_k$ thus $a_k = 0$ for k even. Therefore $p(x)$ is in the span of $\{x, x^3, x^5, ...\}$ hence $A \subseteq \text{span}(\{x, x^3, x^5, ...\})$. On the other hand for any finite linear combination $p(x) = \sum_{k=0}^{n} a_k x^{2k+1}$ we have

$$p(x) = \sum_{k=0}^{n} a_k x^{2k+1} = -\sum_{k=0}^{n} a_k (-x)^{2k+1} = -p(-x)$$

so $\text{span}(\{x, x^3, x^5, ...\}) \subseteq A$ hence $A = \text{span}(\{x, x^3, x^5, ...\})$.

Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial in B then $p(x) = p(-x)$ so

$$\sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} a_k (-x)^k = \sum_{k=0}^{n} (-1)^k a_k x^k$$

Since the two polynomials are equal then $a_k = (-1)^k a_k$ thus $a_k = 0$ for k odd. Therefore $p(x)$ is in the span of $\{1, x^2, x^4, ...\}$ hence $B \subseteq \text{span}(\{1, x^2, x^4, ...\})$. On the other hand for any finite linear combination $p(x) = \sum_{k=0}^{n} a_{2k} x^{2k}$ we have

$$p(x) = \sum_{k=0}^{n} a_{2k} x^{2k} = \sum_{k=0}^{n} a_k (-x)^{2k+1} = p(-x)$$

so $\text{span}(\{1, x^2, x^4, ...\}) \subseteq B$ hence $B = \text{span}(\{1, x^2, x^4, ...\})$. □

Exercise 3.2:24. Determine whether the given subset of $C^1(\mathbb{R})$ consisting of all even functions (functions f such that $f(x) = f(-x)$ for every value of x) is also a subspace.

Solution. Let f and g be even functions, then

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha f(-x) + \beta g(-x) = (\alpha f + \beta g)(-x)$$

so the subset of even function is a subspace. □

Exercise 3.2:25. Let $C[a, b]$ be the vector space of continuous real-valuedfunctions defined on the interval $[a, b]$. Let $C_0[a, b]$ be the set of functions $f \in C[a, b]$ such that $f(a) = f(b) = 0$. Show that $C_0[a, b]$ is a subspace of $C[a, b]$.

Solution. Let $f, g \in C_0[a, b]$ then

$$(\alpha \cdot f + \beta \cdot g)(a) = \alpha \cdot f(a) + \beta \cdot g(a) = \alpha \cdot 0 + \beta \cdot 0 = 0 = \alpha \cdot f(b) + \beta \cdot g(b) = (\alpha \cdot f + \beta \cdot g)(b)$$

so $C_0[a, b]$ is a subspace of $C[a, b]$. □

Exercise 3.2:26. Show that S and T have the same span in $\mathbb{R}^3$ by showing that the vectors in S are in the span of T and vice versa.

$$S = \{(1, 0, 0), (0, 1, 0)\} \qquad T = \{(1, 2, 0), (2, 1, 0)\}$$

Solution. Since $\text{span}(S)$ and $\text{span}(T)$ are vector subspaces, they are closed under linear combinations and scalar multiplications. Therefore if $S \subseteq \text{span}(T)$ then $\text{span}(S) \subseteq \text{span}(T)$. Likewise if $T \subseteq \text{span}(S)$ then $\text{span}(T) \subseteq \text{span}(S)$. Hence it suffices to show that $S \subseteq \text{span}(T)$ and $T \subseteq \text{span}(S)$. Since

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

then $T \subseteq \text{span}(S)$. Likewise

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{2}{3} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{3} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{2}{3} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \frac{1}{3} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

so $S \subseteq \text{span}(T)$. □

Exercise 3.3:6. Each of Exercises 6 to 8 defines a function from $\mathbb{R}^\infty$ to $\mathbb{R}^\infty$, where $\mathbb{R}^\infty$ is the vector space of sequences (x_k), $k = 1, 2, 3, \ldots$ of Example 3 in Section 2. In each case, show that the function is linear and state whether the function is one-to-one (injective) or not. If it is one-to-one then describe its inverse and the domain of the inverse.

$$g(x_1, x_2, x_3, \ldots) = (x_1, 2x_2, 3x_3, \ldots)$$

Solution. The function is bijective. The inverse is given by

$$g^{-1}(x_1, x_2, x_3, \ldots) = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \ldots)$$

□

Exercise 3.3:7.

$$h(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$$

Solution. The function is not injective since

$$f(1, 1, 1, 1, 1, \ldots) = f(0, 1, 1, 1, 1, \ldots)$$

□

Exercise 3.3:8.

$$p(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$$

Solution. The function is injective. It maps onto the subspace

$$V = \{(x_k) | x_1 = 0\}$$

and an inverse is give by $p^{-1} : V \to \mathbb{R}^\infty$

$$p^{-1}(0, x_2, x_3, x_4, \ldots) = (x_2, x_3, x_4, \ldots)$$

□

Exercise 3.3:11. In Exercises 11 and 12, determine the effect on a sequence $(x1, x2, x3, ...)$ of the given combinations of the functions defined in Exercises 6 to 8.

$$g \circ p \quad \text{and} \quad p \circ g$$

Solution.

$$g \circ p(x_1, x_2, x_3, \ldots) = g(0, x_1, x_2, x_3, \ldots) = (0, 2x_1, 3x_2, 4x_3, \ldots)$$
$$p \circ g(x_1, x_2, x_3, \ldots) = p(x_1, 2x_2, 3x_3, 4x_4, \ldots) = (0, 2x_2, 3x_3, 4x_4, \ldots)$$

□

Exercise 3.3:12.

$$h \circ p \quad \text{and} \quad p \circ h$$

Solution.

$$h \circ p(x_1, x_2, x_3, \ldots) = h(0, x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, x_4, \ldots)$$
$$p \circ h(x_1, x_2, x_3, \ldots) = p(x_2, x_3, x_4, \ldots) = (0, x_2, x_3, x_4, \ldots)$$

□

Exercise 3.3:17. Let $D = d/dx$ act as a transformation from $C^1(\mathbb{R})$ to $C(\mathbb{R})$. (a) If $u(x) = 2x^3$, find $(Dx - xD)(u(x))$, where the operator Dx first multiplies by x and then applies D. (b) Show that $Dx - xD = I$, where I is the identity operator defined by $I(u) = u$ for all u. (c) Is $D^2 - x^2$ equal to $(D+x)(D-x)$? To find out, apply both operators to a general function $u(x)$ in $C^2(\mathbb{R})$ and see if you get the same result.

Solution. (a) Let $u(x) = 2x^3$ then

$$(Dx - xD)(u(x)) = D(2x^4) - x \cdot D(2x^3) = 8x^3 - 6x^3 = 2x^3$$

(b) Let $f \in C^1(\mathbb{R})$. By the product rule

$$D(f \cdot x) = \frac{d(f(x) \cdot x}{dx} = \frac{df(x)}{dx} \cdot x + f(x) = D(f) \cdot x + f$$

therefore

$$(Dx - xD)(f) = D(f \cdot x) - x \cdot D(f) = D(f) \cdot x + f - x \cdot D(f) = f = I(f)$$

(c) A computation utilizing part (b) gives

$$\begin{aligned}(D+x) \circ (D-x)(f) = (D+x)(D(f) - f \cdot x) &= D(D(f) - f \cdot x) + x \cdot (D(f) - f \cdot x) \\ &= D^2(f) - D(f \cdot x) + x \cdot D(f) - f \cdot x^2 \\ &= (D^2 - x^2)(f) - (D(f \cdot x) - x \cdot D(f)) \\ &= (D^2 - x^2)(f) - (Dx - xD)(f) \\ &= (D^2 - x^2)(f) - I(f)\end{aligned}$$

Therefore

$$(D+x) \circ (D-x)(f) \neq (D^2 - x^2)(f)$$

□

Exercise 3.3:20. Show that the given function $S : C([0,1]) \to C[0,1]$ given by $S(u(x)) = \int_0^x e^{-t} u(t) dt$ is linear.

Solution. Let $\alpha, \beta \in \mathbb{R}$ and $f, g \in C([0,1])$ then

$$\begin{aligned}S(\alpha \cdot f + \beta \cdot g)(t) &= \int_0^x e^{-t}(\alpha \cdot f(t) + \beta \cdot g(t)) dt \\ &= \int_0^x e^{-t} \alpha f(t) + e^{-t} \beta g(t) dt \\ &= \alpha \int_0^x e^{-t} f(t) dt + \beta \int_0^x e^{-t} g(t) dt \\ &= \alpha S(f) + \beta S(g)\end{aligned}$$

□

Exercise 3.3:22. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be such that $L(1,2) = (2,3)$ and $L(-1,1) = (1,-1)$. Find a vector $\vec{u}$ in $\mathbb{R}^2$ such that $L\vec{u} = (3,7)$.

Solution. Since

$$2\begin{pmatrix}2\\3\end{pmatrix}-\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}3\\7\end{pmatrix}$$

then by linearity

$$\begin{pmatrix}3\\7\end{pmatrix}=L\left(2\begin{pmatrix}1\\2\end{pmatrix}-\begin{pmatrix}-1\\1\end{pmatrix}\right)=L\left(\begin{pmatrix}3\\3\end{pmatrix}\right)$$

so $\vec{u}=(3,3)$. □

Exercise 3.3:24. Let $L: C([0,1]) \to C([0,1])$ be such that $L(1)=1$, $L(x)=x$ and $L(x^2)=x^2+2$. What is $L(2x^2+x-1)$?

Solution.

$$L(2x^2+x-1)=2\cdot L(x^2)+L(x)-L(1)=2(x^2+2)+x-1=2x^2+x+3$$

□

Exercise 3.4:8. In each of Exercises 8 to 10, describe carefully the image of the given transformation F, state whether the function is linear, and if it is linear, describe its kernel (null-space).

$$F: C(\mathbb{R}) \to C(\mathbb{R}) \quad \text{where } F(u)(x)=e^{u(X)}$$

Solution.

$$\operatorname{Im}(F)=\{f\in C(\mathbb{R})|f(x)>0\}$$

The function is not linear; Indeed $F(1+1)=e^2\neq 2e$. □

Exercise 3.4:9.

$$F: C(\mathbb{R}) \to C^1(\mathbb{R}) \quad \text{where } F(u)(x)=\int_0^x e^{-t}u(t)dt$$

Solution. I claim that

$$\operatorname{Im}(F)=\{f\in C^1(\mathbb{R})|f(0)=0\}$$

Indeed since

$$F(u)(0)=\int_0^0 e^{-t}u(t)dt=0$$

then the condition $f(0)=0$ is necessary for f to be in $\operatorname{Im}(F)$. On the other hand given a $f\in C^1(\mathbb{R})$ let $u(x)=f'(x)\cdot e^x$ then by the fundamental theorem of calculus

$$F(u)=\int_0^x f'(t)dt=f(x)-f(0)=f(x)$$

so $f(0)=0$ is also sufficent.

Let $\alpha,\beta\in\mathbb{R}$ and $f,g\in C(\mathbb{R})$ then

$$\begin{aligned}F(\alpha\cdot f+\beta\cdot g)(t)&=\int_0^x e^{-t}(\alpha\cdot f(t)+\beta\cdot g(t))dt\\&=\int_0^x e^{-t}\alpha f(t)+e^{-t}\beta g(t)dt\\&=\alpha\int_0^x e^{-t}f(t)dt+\beta\int_0^x e^{-t}g(t)dt\\&=\alpha F(f)+\beta F(g)\end{aligned}$$

so F is linear. Since $e^{-t}f(t) = 0$ iff $f(t) = 0$ then

$$\text{Ker}(F) = \{f(x) \equiv 0\}$$

□

Exercise 3.4:10.

$$F : C^{(1)}(\mathbb{R}) \to C(\mathbb{R}) \qquad \text{where } F(u)(x) = u'(x) + u(x)$$

Solution.

$$\text{Im}(F) = \{f | \exists u : u'(x) + u(x)\}$$

Let $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^1(\mathbb{R})$ then

$$\begin{aligned} F(\alpha \cdot f + \beta \cdot g)(t) &= \alpha f'(x) + \alpha f(x) + \beta g'(x) + \beta g(x) \\ &= \alpha(f'(x) + f(x)) + \beta(g'(x) + g(x)) \\ &= \alpha F(f) + \beta F(g) \end{aligned}$$

so F is linear. If $f \in \text{Ker}(F)$ then $f'(x) = -f(x)$ hence it is a solution to the differential equation $\frac{dy}{dx} = -y$. By separation of variables

$$\log(y) + c_1 = \int \frac{1}{y} = -\int x dx = -x + c_2$$

The general solution is thus given by $y(x) = Ke^{-x}$ hence

$$\text{Ker}(F) = \{f(x) = Ke^{-x} | K \in \mathbb{R}\}$$

□

Exercise 3.4:11. In Exercises 11 and 12, describe the image and the null-space of the function defined by $f(\vec{x}) = A\vec{x}$ for the given matrix A.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Solution. Since $(1, 0)$ and $(1, 1)$ are linearly independent then

$$\text{Im}(A) = \mathbb{R}^2$$

$$\text{Ker}(A) = \{\vec{0}\}$$

□

Exercise 3.4:12.

$$A = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$$

Solution. Notice how

$$\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}$$

and

$$\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

so $\text{Ker}(A) = \text{span}(\{(3, -1)\})$ and $\text{Im}(A) = \text{span}(\{(2, 1)\})$ □