**Exercise 10.2:22.** Psychological studies of stimulus and response often attempt to treat these as numerical variables, $s$ and $r$ related by an equation $r = f(s)$. It is sometimes hypothesized that $f$ satisfies a differential equation of the form

$$\frac{dr}{ds} = k\frac{r^n}{s} \quad \text{with } k > 0$$

Which of the two hypotheses on the exponent $n$, $n = 0$ or $n = 1$ is consistent with the following values of $(r, s)$:

$$\{(0.5, 1), (1, 2), (3, 6)\}$$

*Solution.* By seperation of variables we have

$$\int \frac{1}{r^n} dr = \int \frac{k}{s} ds$$

so if $n = 1$ we have $\log(r) = \log(s^k) + c$. Therefore

$$r = K \cdot s^k$$

Suppose $\{(0.5, 1), (1, 2), (3, 6)\}$ are all on the graph of some solution, then since $(r, s) = (0.5, 1)$ is on the graph $K = 0.5$. Since $(1, 2)$ is on the graph then $2 = 2^k$ so $k = 1$. Since $6 \cdot 0.5 = 3$ then it follows that $(3, 6)$ is also on the graph and thus $n = 1$ is consistent with the data.

If instead $n = 0$ then $r = \log(s^k) + c$. Suppose $\{(0.5, 1), (1, 2), (3, 6)\}$ are all on the graph of some solution. Since $(r, s) = (0.5, 1)$ is on the graph then $c = 0.5$. Since $(r, s) = (1, 2)$ is on the graph then $k = \frac{1}{2\log(2)}$. Now

$$\frac{\log(6)}{2\log(2)} + \frac{1}{2} = \frac{\log(3) + 1}{2} \neq 3$$

So $n = 0$ is not consistent with the data. $\square$

**Exercise 10.2:26.** Show that the differential equation

$$\frac{dy}{dx} = y + x$$

cannot be written in the form

$$g(y)\frac{dy}{dx} = f(x)$$

and therefore cannot be solved by separation of variables

*Solution.* Suppose $\frac{dy}{dx} = x + y = \frac{f(x)}{g(y)}$ and assume without loss of generality that $x, y > 0$ then

$$\frac{\partial}{\partial y}\left(\frac{\partial \log(f(x)) - \log(g(y))}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{f'(x)}{f(x)}\right) = 0$$

However

$$\frac{\partial}{\partial y}\left(\frac{\partial \log(x + y)}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x + y}\right) = -\frac{1}{(x + y)^2} \neq 0$$

$\square$

**Exercise 10.2:28.** (a) Let $F(x, y)$ be a homogeneous function of degree zero, then the substitution $y = xu$ transforms the differential equation

$$\frac{dy}{dx} = F(x, y)$$

into

$$\frac{du}{dx} = \frac{F(1, u) - u}{x}$$

(b) Show that $F(x, y) = \frac{(x^2+y^2)}{2xy}$ is homogeneous and use the subsitution of part (a) to change the equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

into an equation of the form

$$\frac{du}{dx} = G(x, u)$$

(c) Solve the last differential equation of part (b), and substitute $u$ with $\frac{y}{x}$ in the resulting solution.

*Solution.* (a) let $y = x \cdot u(x)$ then

$$F(x, y) = \frac{dy}{dx} = x\frac{du}{dx} + u(x)$$

since $F$ is homegeneous of degree zero then $F(x, y) = F(x, xu) = F(1, u)$ so

$$\frac{du}{dx} = \frac{F(1, u) - u}{x}$$

(b) Computing

$$F(tx, ty) = \frac{(tx)^2 + (ty)^2}{2txty} = \frac{t^2(x^2 + y^2)}{2t^2xy} == \frac{x^2 + y^2}{2xy}$$

so $F$ is homogeneous of degree zero. So by part (a)

$$\frac{du}{dx} = \frac{\frac{1^2+u^2}{2u} - u}{x} = \frac{1 - u^2}{2ux}$$

so by separation of variables

$$\int \frac{2u}{1 - u^2} du = \int \frac{1}{x} dx$$

let $v = 1 - u^2$ then $dv = -2udu$ so

$$\int \frac{1}{v} dv = -\int \frac{1}{x} dx$$

hence

$$\log(1 - u^2) = \log(v) = \log\left(\frac{1}{x}\right) + c$$

so applying the exponential function on both sides we have

$$1 - u^2 = K \cdot \frac{1}{x}$$

hence

$$u(x) = \sqrt{1 - \frac{K}{x}}$$

Therefore

$$y(x) = x\sqrt{1 - \frac{K}{x}}$$

We verify first note that since $y = x\sqrt{1 - \frac{K}{x}}$ then $\frac{y^2}{x^2} = 1 - \frac{K}{x}$ so

$$K = x - \frac{y^2}{x} = \frac{x^2 - y^2}{x^2}$$

therefore we can verify as follows

$$\begin{aligned}\frac{dy}{dx} &= \sqrt{1 - \frac{K}{x}} + \frac{x}{2\sqrt{1 - \frac{K}{x}}} \cdot \frac{K}{x^2} \\ &= \frac{y}{x} + \frac{K}{2y} \\ &= \frac{y}{x} + \frac{x^2 - y^2}{2xy} \\ &= \frac{2y^2}{2xy} + \frac{x^2 - y^2}{2xy} \\ &= \frac{y^2 + x^2}{2xy}\end{aligned}$$

□

**Exercise 10.3:6.** In Exercises 6 and 7 find the general solution to the differential equation, then find a paticular solution satisfying the initial condition.

$$\frac{dy}{dx} = y + 1, \quad y(0) = 1$$

*Solution.* By seperation of variables

$$\int \frac{1}{1+y} dy = \int 1 dx$$

so $\log(1 + y) = x + c$ hence $1 + y = e^{x+c}$ so the general solution is given by

$$y(x) = Ke^x - 1$$

If $y(0) = 1$ then $K - 1 = 1$ so $K = 2$ hence the particular solution is

$$y(x) = 2e^x - 1$$

□

**Exercise 10.3:7.**

$$2\frac{dy}{dx} = xy \qquad y(1) = 0$$

*Solution.* By separation of variables

$$\int \frac{1}{y}dy = \int \frac{1}{2}xdx$$

so $\log(y) = \frac{x^2}{4} + c$ for some constant $c$, therefore

$$y(x) = e^{\frac{x^2}{4}+c} = Ke^{\frac{x^2}{4}}$$

with $K = e^c$. If $y(1) = 0$ then $K = 0$, thus the specific solution becomes $y(x) \equiv 0$. □

**Exercise 10.3:9.** Salt solution enters a 100-gallon tank of initially pure water from two different sources. One source provides water containing 1 pound of salt per gallon at a rate of 2 gallons per minute. A second source provides 3 gallons of salt solution per minute at a varying concentration $C(t) = 2e^{-2t}$, measured in pounds of salt per gallon. Assume that the contents of the tank are kept thoroughly mixed at all times and that solution is drawn off at a rate of 5 gallons per minute. Find the amount of salt in the tank at an arbitrary time $t > 0$.

*Solution.* First we set up the differential equation

$$\frac{ds}{dt} = 2 + 3 \cdot C(t) - \frac{s}{20} = 2 + 6e^{-2t} - \frac{s}{20}$$

Rewriting this into the form $s' + \frac{1}{20}s = 2 + 6e^{-2t}$ and using the panzer formula

$$\begin{aligned}
s(t) =& e^{\frac{-t}{20}} \left( \int e^{\frac{t}{20}} \left(2 + 6e^{-2t}\right) dt \right) \\
=& e^{\frac{-t}{20}} \left( \int 2e^{\frac{t}{20}} dt + \int 6e^{\frac{-39t}{20}} dt \right) \\
=& e^{\frac{-t}{20}} \left( \frac{2}{20} e^{\frac{t}{20}} - \frac{49 \cdot 6}{25} e^{\frac{-39t}{20}} + c \right) \\
=& \frac{2}{20} - \frac{39 \cdot 6}{20} e^{-2t} + c
\end{aligned}$$

Since $s(0) = 0$ then

$$c = \frac{39 \cdot 6}{20} - \frac{2}{20}$$

so

$$s(t) = \frac{234}{20}\left(1 - e^{-2t}\right)$$

where is is the number amount of salt in the water measured in pounds. □

**Exercise 10.3:16.** Suppose that a metal bar initially at 300 F is immersed in a water bath at 100 F for 30 minutes and then is transferred to another water bath at 50 F. Assume the validity of Newton's law described in Example 5 of the text. (a) What will the temperature of the bar be after an additional 30 minutes, assuming the cooling coefficientfor the iron in water is $k = 0.1$? (b) Suppose that initially the bar is cooled for 30 minutes in air at 100, for which the cooling coefficient is only $k = 0.07$ and is then immersed in water for 30 minutes. What will the temperature of the bar be at the end of the hour?

*Solution.* (a) By Newtons law of cooling

$$\frac{du}{dt} = k(100 - u)$$

By separation of variables

$$\int \frac{1}{u - 100} du = \int -k dt$$

As long as $u > 100$ the right hand side is positive so $\log(u - 100) = -kt + c$ hence

$$u(t) = 100 + Ce^{-kt}$$

Since the initial temperature is 300 F, then $u(0) = 300$ hence $C = 200$. Letting $k = 0.1$ we have

$$u(30) = 100 + 200 \cdot e^{-3}$$

So after 30 minutes the temperature is $100 + 200 \cdot e^{-3}$ which is greater than 50. Let $\tilde{u}$ be a solution to $\frac{d\tilde{u}}{dt} = k(50 - u)$ with $\tilde{u}(0) = 100 + 200 \cdot e^{-3}$. Solving using separation of variables we find

$$\tilde{u}(t) = 50 + Ce^{-kt}$$

Since $\tilde{u}(0) = 100 + 200 \cdot e^{-3}$ then $C = 50 + 200 \cdot e^{-3}$ so after 1 hour the temperature would be

$$\tilde{u}(30) = 50 + \left(50 + 200 \cdot e^{-3}\right) \cdot e^{-3} = 50 + 50 \cdot e^{-3} + 200 \cdot e^{-6} \approx 53$$

(b) After 30 minutes we have $u(30) = 100 + 200 \cdot e^{-0.07 \cdot 30}$ so after 1 hour the temperature would be

$$\tilde{u}(30) = 50 + \left(50 + 200 \cdot e^{-0.07 \cdot 30}\right) \cdot e^{-3} \approx 54$$

□

**Exercise 3.1:2.** Exercises 2 and 4 give information about a linear function $f$. In each case find the matrix $A$ that represents $f$ in the form $f(x) = Ax$ and determine whether the functions one-to-one.

$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} \qquad f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

*Solution.*

$$A = \begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}$$

Since $(2, 1)$ and $(1, 1)$ is obviously linearly independent, then $A$ is injective. □

**Exercise 3.1:4.**

$$f\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} \qquad f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} \qquad f\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

*Solution.*

$$A = \begin{pmatrix}-1 & 1 & 2\\1 & -1 & 1\end{pmatrix}$$

Since $-(1, -1) = (-1, 1)$ then $(-1, 1)$ and $(1, -1)$ are linearly dependent, so $A$ is not injective. $(f(1, -1, 0) = 0 = f(0, 0, 0))$ □

**Exercise 3.1:6.** Exercises 6 and 8 give information about linear functions $f$. In each case find $f(\vec{e}_k)$.

$$f\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix} \qquad f\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$$

*Solution.*

$$f\begin{pmatrix}1\\0\end{pmatrix}=f\left(\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}\right)=\frac{1}{2}\begin{pmatrix}2\\1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix}$$

$$f\begin{pmatrix}0\\1\end{pmatrix}=f\left(\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}\right)=\frac{1}{2}\begin{pmatrix}2\\1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}\frac{3}{2}\\0\end{pmatrix}$$

□

**Exercise 3.1:8.**

$$f\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix} \qquad f\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}0\\1\\1\end{pmatrix} \qquad f\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}2\\0\\0\end{pmatrix}$$

*Solution.*

$$f\begin{pmatrix}1\\0\\0\end{pmatrix}=f\left(\begin{pmatrix}2\\1\\0\end{pmatrix}+\begin{pmatrix}0\\1\\1\end{pmatrix}-\begin{pmatrix}1\\2\\1\end{pmatrix}\right)=\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}2\\0\\0\end{pmatrix}-\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}3\\1\\2\end{pmatrix}$$

$$f\begin{pmatrix}0\\1\\0\end{pmatrix}=f\left(\begin{pmatrix}2\\1\\0\end{pmatrix}-2\begin{pmatrix}1\\0\\0\end{pmatrix}\right)=\begin{pmatrix}1\\2\\3\end{pmatrix}-2\begin{pmatrix}3\\1\\2\end{pmatrix}=\begin{pmatrix}-5\\0\\-1\end{pmatrix}$$

$$f\begin{pmatrix}0\\0\\1\end{pmatrix}=f\left(\begin{pmatrix}0\\1\\1\end{pmatrix}-\begin{pmatrix}0\\1\\0\end{pmatrix}\right)=\begin{pmatrix}2\\0\\0\end{pmatrix}-\begin{pmatrix}-5\\0\\-1\end{pmatrix}=\begin{pmatrix}7\\0\\1\end{pmatrix}$$

□

**Exercise 3.1:12.** Find the matrix that represents the composition $g \circ f$. Also, say what the domain and range of $g \circ f$ are.

$$f(\vec{x})=\begin{pmatrix}1&2&0\\2&2&3\end{pmatrix}\vec{x} \qquad \text{and} \qquad g(\vec{x})=\begin{pmatrix}2&0\\3&-1\end{pmatrix}\vec{x}$$

*Solution.*

$$g\circ f(\vec{x})=\begin{pmatrix}2&4&0\\1&4&-3\end{pmatrix}\vec{x}$$

The domain is $\mathbb{R}^3$ and the image is $\mathbb{R}^2$. □

**Exercise 3.1:17.** (a) Show that

$$U=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix} \qquad \text{and} \qquad V=\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix}$$

represents 90 degrees rotation of $\mathbb{R}^3$ about the $x_1$-axis and $x_2$-axis respectively. Find the matrix $W$ that represents a 90 degree rotation about the $x_3$-axis. Also find $U^{-1}$ and $V^{-1}$ which represent rotations in the opposite direction. (b) Compute $UVU^{-1}$ and $VUV^{-1}$ and interpret the results geometrically by checking out what they do to basis vectors.

*Solution.* (a) First we compute

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}x_1\\ -x_3\\ x_2\end{pmatrix}$$

Since $U$ fixes the first coordinate, this is some linear transformation around the $x_1$-axis. Let $\theta$ denote the angle between $\vec{x}$ and $U\vec{x}$ measured in a plane parallel to $x_2, x_3$ plane. Then

$$\cos(\theta)\cdot(x_2^2 + x_3^2)^2 = \begin{pmatrix}x_2\\ x_3\end{pmatrix} \bullet \begin{pmatrix}-x_3\\ x_2\end{pmatrix} = 0$$

so $U$ is a 90 degree rotation around the $x_1$ axis. For $V$ we likewise compute

$$\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}x_3\\ x_2\\ -x_1\end{pmatrix}$$

Since $V$ fixes the second coordinate, this is a linear transformation around the $x_2$-axis. Let $\theta$ denote the angle between $\vec{x}$ and $V\vec{x}$ measured in a plane parallel to $x_3, x_1$ plane. Then

$$\cos(\theta)\cdot(x_3^2 + x_1^2)^2 = \begin{pmatrix}x_1\\ x_3\end{pmatrix} \bullet \begin{pmatrix}x_3\\ -x_1\end{pmatrix} = 0$$

so $V$ is a 90 degree rotation around the $x_2$ axis. Let

$$W = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Since

$$\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}-x_2\\ x_1\\ x_3\end{pmatrix}$$

Since $W$ fixes the third coordinate, this is a linear transformation around the $x_3$-axis. Let $\theta$ denote the angle between $\vec{x}$ and $W\vec{x}$ measured in a plane parallel to $x_1, x_2$ plane. Then

$$\cos(\theta)\cdot(x_1^2 + x_2^2)^2 = \begin{pmatrix}x_1\\ x_2\end{pmatrix} \bullet \begin{pmatrix}-x_2\\ x_1\end{pmatrix} = 0$$

so $W$ represents a 90 degree rotation around the $x_3$-axis.

$$U^{-1} = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix} \quad \text{and} \quad V^{-1} = \begin{pmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$$

(b)

$$UVU^{-1} = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} = W \quad \text{and} \quad VUV^{-1} = \begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} = W^{-1}$$

The geometric meaning is that first rotating 90 degrees clockwise around the $x_1$ axis, then rotating 90 degrees counter clockwise around the $x_2$ axis and then rotating 90 degrees counterclockwise around the $x_1$ axis is the same rotating 90 degrees around the $x_3$ axis. □

**Exercise 3.1:19.** Let $\vec{n}$ be the unit vector $\left(\frac{3}{7}, \frac{6}{7}, \frac{2}{6}\right)$, and let $P_{\vec{n}} : \mathbb{R}^3 \to \mathbb{R}^3$ be the associated projection function as in Example 6. Find the matrix of $P_{\vec{n}}$ by finding the image of each of the standard basis vectors under it.

*Solution.* Let $\vec{e}_1, \vec{e}_2, \vec{e}_3$ denote the standard basis vectors, then

$$P_{\vec{n}}(\vec{e}_1) = (\vec{e}_1 \bullet \vec{n}) \cdot \vec{n} = \frac{3}{7} \cdot \vec{n} = \begin{pmatrix} \frac{9}{49} \\ \frac{18}{49} \\ \frac{6}{49} \end{pmatrix}$$

$$P_{\vec{n}}(\vec{e}_2) = (\vec{e}_2 \bullet \vec{n}) \cdot \vec{n} = \frac{6}{7} \cdot \vec{n} = \begin{pmatrix} \frac{18}{49} \\ \frac{36}{49} \\ \frac{12}{49} \end{pmatrix}$$

$$P_{\vec{n}}(\vec{e}_3) = (\vec{e}_3 \bullet \vec{n}) \cdot \vec{n} = \frac{2}{7} \cdot \vec{n} = \begin{pmatrix} \frac{6}{49} \\ \frac{12}{49} \\ \frac{4}{49} \end{pmatrix}$$

So the matrix of $P_{\vec{n}}$ is given by

$$A = \begin{pmatrix} | & | & | \\ P_{\vec{n}}(\vec{e}_1) & P_{\vec{n}}(\vec{e}_2) & P_{\vec{n}}(\vec{e}_3) \\ | & | & | \end{pmatrix} = \begin{pmatrix} \frac{9}{49} & \frac{18}{49} & \frac{6}{49} \\ \frac{18}{49} & \frac{36}{49} & \frac{12}{49} \\ \frac{6}{49} & \frac{12}{49} & \frac{4}{49} \end{pmatrix}$$

□