Exercise 1. Find $\exp(tA)$ by computing the succesddive terms, $I, tA, t^2A/2, \ldots$ in the series definition where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

solution. Clearly $A^2 = I$ so $A^{2n} = I$ and $A^{2n+1} = A$. Therefore

$$\exp(tA) = \sum_{n=0}^{\infty} \frac{t^{2n} I}{2n!} + \sum_{n=0}^{\infty} \frac{t^{2n+1} A}{(2n+1)!} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{t^{2n}}{2n!} & \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \\ \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} & \sum_{n=0}^{\infty} \frac{t^{2n}}{2n!} \end{pmatrix} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$$

□

Exercise 2. Find, using the nilpotent-diagonal decomposition, the exponential of the matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & 1 \\ -1 & 0 & 4 \end{pmatrix}$$

Solution. By inspection e_2 and $e_1 + e_3$ are both eigenvectors for the eigenvalue 3. Since $\text{Tr}(A) = 9$ the last eigenvalue is also 3. Take $N = A - 3I$. Then $N^2 = 0$ so

$$\exp(A) = \exp(3I + N) = \exp(3I)\exp(N) = \exp(3I)(I + N) = \exp(3)I + \exp(3)N$$

Each term is now easy to compute

$$\exp(A) = \begin{pmatrix} e^3 & 0 & 0 \\ 0 & e^3 & 0 \\ 0 & 0 & e^3 \end{pmatrix} + \begin{pmatrix} -e^3 & 0 & e^3 \\ -e^3 & 0 & e^3 \\ -e^3 & 0 & e^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & e^3 \\ -e^3 & e^3 & e^3 \\ -e^3 & 0 & 2e^3 \end{pmatrix}$$

□

Exercise 4. Let (X, d) be any metric space. Show that

$$\tilde{d} = \frac{d(a,b)}{1 + d(a,b)}$$

defines a metric on X. Give a bound on the maximum distance between any elements of X with respect to the metric $\tilde{d}$.

Solution. Positivity and symmetry is immediate. It remains to show the triangle inequality. Consider $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = \frac{x}{1+x}$. Then $f'(x) = \frac{1}{(1+x)^2} > 0$ so f in increasing and $f''(x) = \frac{-2}{(1+x)^3} < 0$ so f is concave. Therefore if $x \le y + z$ then

$$f(x) \le f(x + y) \le f(y) + f(z)$$

Applying this with $x = d(a,b)$, $y = d(a,c)$ and $z = d(c,b)$ deduces the triangle inequality for $\tilde{d}$ from the triangle inequality for d. □

Exercise 5. Consider the metric space $(\mathbb{R}^n, d)$ where $d(x,y) = ||x - y||$ Show that this in fact is a metric space.

Solution. Positivity and symmetry is immediate. It remains to show the triangle inequality. It suffices to show that

$$||u+v|| \leq ||u|| + ||v||$$

since then

$$d(x,z) = ||x-z|| = ||x-y+y-z|| \leq ||x-y|| + ||y-z|| = d(x,y) + d(y,z)$$

Consider

$$||u+v||^2 = \langle u, v\rangle = \langle u, u\rangle + \langle v, v\rangle + 2\langle u, v\rangle \leq ||u||^2 + ||v||^2 + 2||u||||v|| = (||u|| + ||v||)^2$$

Applying the square root function, which is an order preserving function, gives the desired result. □

Exercise 6. Consider the metric space $(\mathbb{R}, d)$ given by $d(x,y) = |e^{-x} - e^{-y}|$. Given an example of a Cauchy sequence that does not converge and a heuristic justification.

Solution. $x \mapsto -\log(x)$ is an isometry from $((0,\infty), d_{std})$ to $((-\infty,\infty), d)$. So take any sequence, say $\frac{1}{n}$ on $((0,\infty), d_{std})$ converging to 0 in $(\mathbb{R}, d_{std})$. Since the sequence converges in $(\mathbb{R}, d_{std})$, then the sequence is Cauchy, and since $x \mapsto -\log(x)$ is an isometry then the sequence

$$\{\log(n)\}$$

is also Cauchy. However this sequence clearly does not converge. □