Exercise 10.1A:2. By substituting into the given differential equation in Exercises 2 and 3, verify that the corresponding formula to the right gives one or more solutions to the differential equation. Then determine the arbitrary constant so that the differentiable function $y(x)$ satifies the given initial condition of the form $y(a) = b$ and satisfies the given differential equation on an interval containing a.

$$\frac{dy}{dx} = -\frac{x}{y}; \quad y = \sqrt{a^2 - x^2}, \quad |x| < a, \quad y(1) = 4$$

Solution. Let $y(x) = \sqrt{a^2 - x^2}$ then

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{a^2 - x^2}} = \frac{x}{y(x)}$$

Since $4 = y(1) = \sqrt{a^2 - 1^2}$ then

$$a^2 = 4^2 + 1 = 17$$

hence $a = \sqrt{17}$. □

Exercise 10.1A:3.

$$y' + y = 0; \quad y = Ke^{-x}, \quad y(5) = 6$$

Solution. Let $y(x) = Ke^{-x}$ then $y'(x) = -Ke^{-x}$ so

$$y' + y = -Ke^{-x} + Ke^{-x} = 0$$

Since $y(5) = 6$ then $6 = Ke^{-5}$ so

$$K = 6e^5$$

□

Exercise 10.1A:7. For each of the differential equations in 7 and 8 of the form $y' = F(x, y)$, sketch the associated direction field, locating a short segment with slope $F(x, y)$ at enough points (x, y) so that a geometric pattern begins to appear. Then sketch into the same picture a solution graph containing the given point (x_0, y_0).

$$y' = \frac{y}{x}, \quad (x_0, y_0) = (1, 2)$$

Exercise 10.1A:8.

$$\frac{dy}{dx} = -\frac{x}{y}, \quad (x_0, y_0) = (1, 1)$$

Exercise 10.1A:11. An isocline in a direction field is a curve along which the directions of the field are all the same. Finding the isoclines of a field is helpful in sketching the field because the direction segments on an isocline are all parallel. For the direction field determined by a differential equation $y' = F(x, y)$, the isoclines satisfy equations of the form $F(x, y) = m$, where m is some constant slope. In exercise 11, sketch several isoclines, and then sketch the direction field by drawing parallel segments crossing the isocline curves $F(x, y) = m$ with slope m.

$$y' = -\frac{y}{x}$$

Exercise 10.1A:20. The differential equation is of the special form $y' = f(x)$, having isoclines that are lines parallel to the y-axis. Thus to sketch the direction field you need to determine only one slope on each such line, making all slope-segments centered on that line parallel to the first one. Sketch the direction field for each of the following differential equations and then use the field to sketch in a few solution graph

$$y' = x^4$$

Exercise 10.1A:25. The differential equation $y' = \sqrt{1-y^2}$ is satisfied by $y(x) = \sin(x+a)$ on any interval on which $y'(x) \geq 0$. The differential equation is also satisfied by $y(x) = 1$ and $y(x) = -1$. Show that on the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ there are infinitely many different solutions passing through $(0,1)$ and also infinitely many different solutions passing through $(0,-1)$. Explain why the uniqueness part of Theorem 1.1 (in the textbook) is not contradicted by this example

Solution. For every $a \in (-\pi, \pi)$ define a function $f_a : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (-1,1)$ by

$$f_a = \begin{cases} \sin(x+a) & \text{for } -\frac{\pi}{2} - a < x < \frac{\pi}{2} - a \\ 1 & \text{for } x \geq \frac{\pi}{2} - a \\ -1 & \text{for } x \leq -\frac{\pi}{2} - a \end{cases}$$

Then clearly f_a is continuous everywhere and f_a is differentiable everywhere except possibly at $\frac{\pi}{2} - a$ and $-\frac{\pi}{2} - a$. Recall that by definition f_a is differentiable at p whenever the limit

$$\lim_{h \to 0} \frac{f_a(p+h) - f_a(p)}{h}$$

exists. Taking $p = \frac{\pi}{2} - a$ and approaching from the left we have

$$\lim_{h \to 0^-} \frac{f_a(\frac{\pi}{2} - a + h) - f_a(\frac{\pi}{2} - a)}{h} = \lim_{h \to 0^-} \frac{\sin(\frac{\pi}{2} + h) - 1}{h} = 0$$

Approaching from the right we have

$$\lim_{h \to 0^+} \frac{f_a(\frac{\pi}{2} - a + h) - f_a(\frac{\pi}{2} - a)}{h} = \lim_{h \to 0^+} \frac{1-1}{h} = 0$$

Since both the left and right limit exists, then the limit exists hence f_a is differentiable at $p = \frac{\pi}{2} - a$ with derivative $f_a'(\frac{\pi}{2} - a) = 0$.

Similarly for $p = -\frac{\pi}{2} - a$ we have

$$\lim_{h \to 0^+} \frac{f_a(-\frac{\pi}{2} - a + h) - f_a(-\frac{\pi}{2} - a)}{h} = \lim_{h \to 0^+} \frac{\sin(-\frac{\pi}{2} + h) - 1}{h} = 0$$

Approaching from the right we have

$$\lim_{h \to 0^-} \frac{f_a(-\frac{\pi}{2} - a + h) - f_a(-\frac{\pi}{2} - a)}{h} = \lim_{h \to 0^-} \frac{-1+1}{h} = 0$$

so the limit exists hence f_a is differentiable at $-\frac{\pi}{2} - a$ with derivative $f_a'(-\frac{\pi}{2} - a) = 0$.

I claim that for each $a \in (-\pi, \pi)$ the function f_a is a distinct solution to the differential equation $f_a'(x) = \sqrt{1 - (f_a(x))^2}$. If $a \geq \frac{\pi}{2}$ then $f_a(0) = 1$ so this would show that there are infinitely many different solutions passing through $(0,1)$. Similarly if $a \leq -\frac{\pi}{2}$ then $f_a(0) = -1$

so this would show that there are infinitely many different solutions passing through $(0, -1)$. To prove the claim suppose first that $-a - \frac{\pi}{2} < x < \frac{\pi}{2} - a$ then we have $f_a = \sin(x + a)$ and

$$\frac{df_a}{dx} = \frac{d}{dx}\sin(x+a) = \cos(x+a) \quad \text{for } -a - \frac{\pi}{2} < x < \frac{\pi}{2} - a$$

Since $\cos(x - a)$ is positive for $-a - \frac{\pi}{2} < x < \frac{\pi}{2} - a$ then

$$\frac{df_a}{dx} = \cos(x+a) = \sqrt{1 - (\sin(x+a))^2} = \sqrt{1 - (f_a(x))^2} \text{ for } -\frac{\pi}{2} - a < x < \frac{\pi}{2} - a$$

Suppose now that $x \leq -\frac{\pi}{2} - a$ then we have $f_a = -1$ so

$$\frac{df_a}{dx} = 0 = \sqrt{1 - (f_a(x))^2} \quad \text{for } x \leq -\frac{\pi}{2} - a$$

Similarly if $x \geq \frac{\pi}{2} - a$ then $f_a = 1$ so

$$\frac{df_a}{dx} = 0 = \sqrt{1 - (f_a(x))^2} \quad \text{for } x \geq \frac{\pi}{2} - a$$

To check that distinct values of $a \in (-\pi, \pi)$ give rise to distinct functions simply note that $f_a'(x) > 0$ if and only if $-a - \frac{\pi}{2} < x < \frac{\pi}{2} - a$ so the domain on which the function is increasing is distinct for each $a \in (-\pi, \pi)$.

The uniqueness part of theorem 1.1 is not violated, since $y \mapsto \sqrt{1 - y^2}$ is not differentiable at 1. In fact, there is no extension of $y \mapsto \sqrt{1 - y^2}$ to some neighbourhood of 1 which is even Lipschitz near 1 (The derivative goes to infinity as y approaches 1). □

Exercise 10.1B:1. In Exercises 1 to 10, solve each differential equation by direct integration, and find the particular solution that satisfies the associated initial condition by determining one or more constants of integration.

$$y' = x(1-x), \quad y(0) = 1$$

Exercise 10.1B:4.

$$\frac{du}{dv} = v^2 + 1, \quad u(-1) = 1$$

Exercise 10.1B:5.

$$y'' = 1, \quad y(0) = 1, \quad y'(0) = 1$$

Exercise 10.1B:10.

$$y'''' = x, \quad y(0) = y''(0) = 0, \quad y'(1) = y'''(1) = 1$$

Exercise 10.2:4. n Exercises 4 and 9, solve each differential equation by direct integration, and find the particular solution that satisfies the associated initial condition by determining one or more constants of integration

$$\frac{du}{dv} = v^2 + 1, \quad u(-1) = 1$$

Exercise 10.2:9.

$$\frac{d^2x}{dt^2} = e^t, \quad x(0) = 1, \quad \left.\frac{dx}{dt}\right|_{t=0} = 0$$

Exercise 10.2:18. In 18 and 19. Find a solution formula for the differential equations, and then find a particular solution that satisfies the given additional condition. Verify by substitution that your solution does satisfy the differential equation.

$$\frac{dy}{dt} = 2ty, \quad y(0) = 2$$

Exercise 10.2:19.

$$y' = \frac{x}{y^2}, \quad y(1) = 0$$